Almost global attraction in planar systems

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Abstract

In this work, we present a result relating the recent ideas of almost global stability and density functions with the classical Poincaré–Bendixson theory for planar systems.

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1. Introduction

The almost global stability of dynamical systems is a concept weaker than global asymptotical stability but that can fit well in nonlinear control applications, specially when it is combined with local asymptotical stability. The concept and a sufficient condition for almost global stability were stated in the year 2001 by Anders Rantzer [10] as a dual Lyapunov method and has opened a new research direction in the nonlinear control field for both analysis and synthesis. The main result in [10] is based on the existence of a density function, a kind of a dual of a Lyapunov function, that allows us to measure the growth of given sets along the flow. Some works have explored the analysis and synthesis of nonlinear control systems [1,2,7–9]; converse results were proved in [4,11]; in [5] we have stated some relationships between almost global stability and monotone measures.

In this work, we explore the conditions that the existence of monotone measures imposes to the behavior of two-dimensional vector fields, blending the new ideas of almost global attraction with the classical result of Poincaré–Bendixson. We think that these results can help in understanding density functions and almost global stability.

In Section 2, we recall the basic definitions of almost global stability and density functions and the results between both concepts. We also briefly introduce the idea of monotone measure. In Section 3, we state and prove the main result for planar systems. Finally we present a counterexample in dimension three and some conclusions.

2. Preliminaries

In this section, we introduce the works reported in [10]. We say that the origin is an almost global...
stable (a.g.s.) equilibrium point if the complement of the set of points that are attracted to the origin has zero Lebesgue measure. For \( x_0 \in \mathbb{R}^n \), let \( \Phi(t, x_0) \) denote the time \( t \) of the trajectory that starts at \( x_0 \). Denote by \( R \) the set of initial conditions whose trajectories converge to the origin:

\[
R = \left\{ x \in \mathbb{R}^n \mid \lim_{t \to +\infty} \Phi(t, x) = 0 \right\}.
\]

Then the system is a.g.s. if the set \( R^C \), the complement of \( R \), has zero Lebesgue measure. This concept of stability is weaker than the classical global asymptotic stability (g.a.s.) but can complement well the local asymptotical stability (l.a.s.) property (i.e. when only local asymptotical behavior can be stated). The key contribution of [10] was the introduction of a particular kind of functions that for a.g.s. systems play a role similar to the Lyapunov functions for asymptotically stable systems: the density functions. Given a dynamical system \( \dot{x} = f(x) \), a density function for this system is a scalar function \( \rho : \mathbb{R}^n \setminus \{0\} \to [0, +\infty) \), of class \( C^1 \), integrable outside of a ball centered at the origin, and such that the following divergence condition is satisfied:

\[
\nabla \cdot (\rho f)(x) > 0 \text{ almost everywhere (a.e.)} \tag{1}
\]

The main result in [10] says that the existence of a density function implies the almost global stability of the origin.

A density function gives us a system-related way of measure sets in \( \mathbb{R}^n \). We can construct a Borel measure \( \mu \) for the state space \( \mathbb{R}^n \) that grows along the trajectories. For every set \( Z \), we put

\[
\mu(Z) = \int_Z \rho(x) \, dx.
\]

If \( 0 < \mu(Z) < \infty \), the divergence sign condition implies that for every positive time \( t \) we have

\[
\mu(f^t(Z)) > \mu(Z)
\]

(see Lemma A.1 in [10]). This measure \( \mu \) also satisfies that \( \mu(B^t(0, \epsilon)) \leq C \) for every \( \epsilon > 0 \). We refer to this as a monotone measure bounded at infinity. The existence of a measure like that is a necessary and a sufficient condition for some kind of almost global stable systems, as was studied in [5].

3. The main result

We present here a result for two-dimensional spaces that relates non-preserving measures (and hence monotone measures) with the Poincaré–Bendixson theory. First of all we recall the definition of the \( \omega \) and \( \alpha \) limit sets for a given trajectory.

**Definition 3.1.** For \( x \in \mathbb{R}^n \), we define the \( \omega \)-limit of \( x \) as the set

\[
\omega(x) = \left\{ y \in \mathbb{R}^n \mid \exists (t_n) \text{ with } \lim_{n \to +\infty} t_n = +\infty, \right. \]

\[
\left. \lim_{n \to +\infty} \Phi(t_n, x) = y \right\}.
\]

The \( \alpha \)-limit is defined in the same way with \( t_n \to -\infty \).

It can be proved that if the trajectory \( \{\Phi(t, x)\} \) is bounded for \( t \to \pm \infty \), \( \omega(x) \) (\( \alpha(x) \)) is a non-empty, compact, connected and invariant set [3,12] and we can talk of the \( \omega(\alpha) \)-limit of the whole trajectory through \( x \). Now we recall the classical result of Poincaré–Bendixson [6]:

**Theorem 3.1.** Consider the system \( \dot{x} = f(x) \) where \( f \in C^1(\mathbb{R}^2, \mathbb{R}^2) \). Let \( M \subset \mathbb{R}^2 \) be a compact, positive invariant set, containing a finite number of equilibrium points. Then, if \( x \in M \), \( \omega(x) \) can be

- a singular point,
- a closed orbit,
- singular points \( p_1, p_2, \ldots, p_n \) and regular orbits \( \gamma \) such that \( \alpha(\gamma) = p_i, \omega(\gamma) = p_j, i, j = 1, \ldots, n. \)

The main result of the paper is the following.

**Theorem 3.2.** Consider the system

\[
\dot{x} = f(x).
\]

where \( f \in C^1(\mathbb{R}^2, \mathbb{R}^2) \). Assume that \( f(0) = 0 \) and that there is a finite number of equilibrium points in any compact set of \( \mathbb{R}^2 \). Suppose there exists a measure \( \mu \leq m \), with \( m \) the Lebesgue measure, satisfying that for every bounded and measurable set \( Y, \mu(Y) < \infty \), and if \( 0 < \mu(Y) < \infty \), there exists \( t \neq 0 \) such that

\[
\mu\left[f^t(Y) \right] \neq \mu(Y).
\]

(2)
Then, if the set of initial conditions attracted by the origin is dense\(^1\) in \(\mathbb{R}^2\), the origin is a locally asymptotically stable equilibrium point.

The following lemma will be used later in the proof of Theorem 3.2.

**Lemma 3.1.** In the hypothesis of Theorem 3.2, the only possible non-empty \(\omega\)-limit sets are equilibrium points.

**Proof.** Consider a point \(x \in \mathbb{R}^2\) with non-empty \(\omega\)-limit. As in Theorem 3.1, \(\omega(x)\) can only be

1. a singular point,
2. a closed orbit,
3. singular points \(p_1, p_2, \ldots, p_n\) and regular orbits \(\gamma\) such that \(x(\gamma) = p_i, \omega(\gamma) = p_j, i, j = 1, \ldots, n\).

The same result follows for the \(\alpha\)-limit set. Typical non-empty \(\omega\)-(\(\alpha\))-limit possible sets for a point \(x\) are shown in Fig. 1. The hypothesis (2) about \(\mu\) implies that the only possible situation for a non-empty \(\omega\)-limit (\(\alpha\)-limit) set is a single equilibrium point, case (d) in Fig. 1, since cases (a), (b) and (c) contain an invariant non-zero Lebesgue measure set and this cannot happen by (2). \(\square\)

**Proof of Theorem 3.2.** Since we have almost global attraction of the origin, we only need to prove local stability. We will do that by contradiction. Suppose that the origin is not a locally stable equilibrium point. Then, there is an \(\varepsilon > 0\), small enough to ensure that \(x = 0\) is the only singular point inside the open ball \(B(0, \varepsilon)\), such that for every non-zero \(n \in \mathcal{N}\) we can find an \(x_n \in \mathbb{R}^2\) with

\[
\|x_n\| < \frac{1}{n}, \quad \sup_{t \geq 0} \|f^t(x_n)\| > \varepsilon
\]

as in Fig. 2. Define \(z_n\) as the first positive intersection of the trajectory \(\{f^t(x_n)\}\) with the sphere \(S(0, \varepsilon)\). Then we obtain a sequence \(\{z_n\}_{n \in \mathcal{N}}\) of points with norm equal to \(\varepsilon\), whose trajectories to the past come close to the origin. Since \(S(0, \varepsilon)\) is a compact set, we can find a sub-sequence, which we still call \(\{z_n\}\), converging to a point \(z \in S(0, \varepsilon)\). We affirm that

\(\omega(z) = \{0\}\).

If it is not the case, there is a positive real \(a\) such that the trajectory never goes inside the ball \(B(0, a)\). Then, since \(x = 0\) is the only singular point in \(B(0, \varepsilon)\), the trajectory \(\{f^t(z)\}\) must leave the ball \(B(0, \varepsilon)\). The situation is shown in Fig. 3.

Then every trajectory starting close enough to \(z\) will accompany \(\{f^t(z)\}\) to the past out of the ball and given a negative time \(t\), there exists a non-zero natural \(N_1\) such that for every \(n > N_1\), \(\{f^t(z_n)\}\) lies outside

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\(^1\) Here, and in the rest of the article, dense is used in the topological sense.
the ball. On the other hand, there is a non-zero natural $N_2$ such that for every $n > N_2$

$$\inf_{t \leq 0} \{ \| f^t(z_n) \| \} < \frac{1}{n}.$$  

Then, for every $n > \max\{N_1, N_2\}$, the negative trajectory through $z$ leaves $B(0, \epsilon)$ before it gets close to the origin, but this cannot occur since $z_n$ was defined in a way such that the piece of trajectory from $x_n$ to $z_n$ is totally in the inside of the closed ball $B(0, \epsilon)$. Then $z(z) = 0$.

Now consider a transversal section $L$ to the trajectory through $z$, that is, a closed line segment containing no equilibrium points and such that at every point the field $f$ is not parallel to the direction of $L$. On this section $L$, we can find a point $y_0$, arbitrarily close to $z$, whose $\omega$-limit is the origin, since this kind of trajectories are dense in the plane. Then, as in the Poincaré–Bendixson theorem, we can construct a closed path with the negative trajectory through $z$, a piece of the transversal section $L$ and the positive trajectory through $y_0$. This path limits a closed region of the plane, with a finite number of equilibrium points inside it. The first situation we can have is the one shown in Fig. 4(a). On the transversal section, we can find two points whose $\omega$-limit sets are the origin and their $\alpha$-limit sets are some singular point (could be other than the origin). The trajectories through these points are like the bold ones in Fig. 4(a). The other case is shown in 4(b) and the result is the same as case (a). In both situations, the sets limited by the bold trajectories are invariant and have non-zero Lebesgue measure. This is absurd and then the origin is locally stable equilibrium point.

Remarks.

- If the measure $\mu$ is monotone, then condition (2) is fulfilled for every set with finite $\mu$-measure.
- If the origin is almost global stable, then the density of the attracted trajectories is fulfilled.

We apply the previous result in order to characterize the behavior at infinity of an almost globally stable system.

**Theorem 3.3.** Consider the complete nonlinear system $\dot{x} = f(x)$ with $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. Assume that the set $f^{-1}(\{0\})$ is finite in $\mathbb{R}^2$ and that there is a monotone measure $\mu$ bounded at infinity. If the set

$$A = \{ x \in \mathbb{R}^2 \mid \lim_{t \to +\infty} \| f^{-t}(x) \| = +\infty \}$$

is dense in $\mathbb{R}^2$ then $\infty$ is a locally asymptotically stable point to the past.

**Proof.** We have to show that given an arbitrary positive number $M$, there is a positive number $K$, depending on $M$, such that

$$\text{if } \|x\| > K \Rightarrow \| f^{-t}(x) \| > M \quad \forall t \geq 0$$

and that $K$ can be chosen such that $\| f^{-t}(x) \| \to +\infty$.

Instead of that, we will compactify the plane using the stereographic projection in order to work on the compact Riemann sphere.Doing this, we obtain a dynamical system on the sphere with an a.g.s. equilibrium point at the south pole $S$ (corresponding to the origin of the plane) and an equilibrium point at the north pole $N$ (corresponding to the infinity of the plane). We know that $N$ attracts a dense set of initial conditions to the past and that we can define a Borel measure $\mu$ on the sphere in a way that given any non-zero Lebesgue measure neighborhood $Y$ of $N$ with $S \notin \bar{Y}$, it verifies $0 < \mu(Y) < \infty$ and for every $t > 0$,

$$\mu( f^t(Y) ) > \mu(Y).$$

Then we consider the reversed system

$$\dot{x} = - f(x)$$

on the sphere and we obtain that $N$ attracts a dense set of initial conditions. We can reconstruct the proof of Theorem 3.2, denying the thesis and getting the existence of the bold trajectories of Fig. 4. If the set

\[\text{For details of this construction, see [3, Chapter 7].}\]
A enclosed by this curve has finite measure $\mu$, it is absurd, just as in the previous proof. So, the question we must answer is if $S \in \bar{A}$. But if it was the case, $S$ would be the $x$-limit of the bold trajectories and then $S$ could not attract almost all the initial conditions of the original system. In order to see this, consider again the closed path constructed with the negative trajectory of $z$, the positive trajectory of $y_0$ and a piece of the transversal section through $z$. We draw again the picture in Fig. 5.

Then all the trajectories started outside this closed path must enter it to reach $S$ and these can be done only through the piece of transversal section, which can be made arbitrarily small because of the dense assumption on the set of initial conditions attracted by the north pole $N$ to the past.

□

The counter-reciprocal version of the previous theorem is very interesting.

**Corollary 3.3.1.** Consider the nonlinear system $\dot{x} = f(x)$ with $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. Assume that the set $f^{-1}(\{0\})$ is finite in $\mathbb{R}^2$ and that there is a monotone Borel measure $\mu$ bounded at infinity. If there is at least one trajectory that goes to infinity to the future, then the set of initial conditions whose trajectories go to infinity to the past is not dense in $\mathbb{R}^2$.

**Example 3.1 (Rantzer [10]).** Consider the planar system

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
-2x_1 + x_1^3 - x_2^2 \\
-6x_2 + 2x_1x_2
\end{bmatrix}.
$$

It has four equilibria at $(0, 0)$, $(2, 0)$ and $(3, \pm \sqrt{3})$. We note that the axis $\{x_2 = 0\}$ is an invariant set. Then, if we consider the initial condition $(x_{10}, 0)$ with $x_{10} > 2$, we find out that the trajectory goes to infinity. Besides that, the system admits a density function

$$
\rho(x_1, x_2) = [x_1^2 + x_2^2]^{-2},
$$

$$
\nabla \cdot (\rho f)(x_1, x_2) = 16x_2^2 \cdot [x_1^2 + x_2^2]^{-4}.
$$

Observe that the local stability of the origin and the existence of the monotone Borel measure bounded at infinity prevent the existence of limit cycles. Then, we can conclude that there exist trajectories that do not go to infinity to the past and then they must go to another equilibrium point.

Of course the previous result of Example 3.1 could have been obtained through other ways. For example, we can classify the equilibrium points and realize that the only divergent trajectory is the right half-line we have previously found. Moreover, the initial conditions that are not attracted by the origin are located on this right half-line and on the stable manifold of $(2, 0)$, as can be seen in Fig. 6.

The main result of this section is deeply grounded in the topological consequences of the dimension 2. The following example shows that in dimension 3 we can have a measure satisfying (2), but the origin can be a.g.s. and not locally stable.
Example 3.2. Consider the following dynamical system defined on $\mathbb{R}^3$:

\[
\begin{align*}
\dot{x} &= x^2 - y^2, \\
\dot{y} &= 2xy, \\
\dot{z} &= -z.
\end{align*}
\]

In the $z$ direction we have the decoupled dynamic

\[ z(t) = e^{-t} \cdot z_0, \]

and at the plane $z=0$ the dynamic has the phase portrait depicted in Fig. 7. As can be proved analytically, the trajectories on $z=0$ are circumferences with the center on the line $x = 0$. So the origin $(0, 0, 0)$ is an almost globally stable equilibrium point but not locally stable.

4. Conclusions

We have presented a result for planar systems that states a relationship between the new concepts of almost global stability and density functions and the classical theorem of Poincaré–Bendixson. We have also shown a counter-example in $\mathbb{R}^3$ in order to emphasize that the result is based on the topological structure of two-dimensional systems.

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References