Robust Synthesis of Dynamic Prefilters

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\textbf{Abstract}

The design of a dynamic prefilter from the point of view of robust performance under structured uncertainty is considered. Synthesis methods are presented for the cases of $H_\infty$ or $H_2$ performance, and Linear Time-Invariant (LTI) or Linear Time Varying (LTV) uncertainty, all of which result in convex optimization problems across frequency. Methods to obtain finite-dimensional approximations are discussed. In the case of $H_\infty$ performance and LTV uncertainty, the synthesis problem is reduced exactly to a finite dimensional Linear Matrix Inequality (LMI) in state space.

\textbf{1 Introduction}

One of the central problems in robust control theory is the design of a controller in the presence of structured uncertainty (see \cite{4,12}). The theory has successfully addressed the analysis of robust performance of a given controller \cite{11,12,13,19,17}. Unfortunately, the robust controller synthesis problem is in an intrinsically harder complexity class, and remain unsolved. For this reason, research has focused on special cases of robust synthesis (see e.g. \cite{3,18}), which not only provide design methods for special problems, but can also be useful for attacking the general problem. This paper is motivated by the search for tractable robust design problems, and by research on two-degree-of-freedom controllers \cite{6,8,9,10} for tracking. Such configuration is depicted in Figure 1, where $r$ denotes the reference input, $s$ the output which must track $r$, $w$ an external disturbance, and $\Delta$ the uncertainty. $P$ and $F$ could be lumped into a controller $K$ and synthesized by any number of techniques. The roles of $P$ and $F$ are somewhat distinct, however, $F$ being the only part which affects stability and disturbance rejection, and $P$ having strong influence in command response. This suggests the possibility of a two-stage design, with different types of performance specifications at each stage. In the nominal case, it was shown in \cite{16} that such strategy is not restrictive: the two designs are independent in the sense that achievable closed loop transfer functions from $r$ to $s$ are not restricted by the choice of a stabilizing $F$. Thus one can design $F$ first for stability and disturbance rejection by, e.g., $H_\infty$ control, and subsequently design $P$ for tracking performance using other criteria ($H_2$, channel decoupling, etc.).

Clearly, independence does not hold for the uncertain case, and in general the choice of $F$ will impose constraints on the achievable tracking performance. Still, since the joint synthesis problem is hard it may be useful to break down the design in this way, allowing also for different kinds of requirements at each stage. An example of this approach is given in \cite{10}.

We will consider the case of structured uncertainty

\begin{equation}
\Delta = \text{diag}[\delta_1 I_{L_1}, \ldots, \delta_L I_{L_L}, \Delta_{L+1}, \ldots, \Delta_{L+F}],
\end{equation}

where the blocks are dynamic operators, and can be of the LTI or LTV type. We assume that $F$ has been already designed and is robustly stabilizing, and focus on the design of a prefilter $P$ for robust tracking, for both the $H_\infty$ and $H_2$ performance criteria. Convex conditions are derived, which extend the work in \cite{7}, where only static prefilters were considered.

While the main tool for canceling uncertainty will be the feedback $F$, the present results are still useful since $P$ can have a significant effect on tracking performance. Since its design is computationally tractable, there is no added difficulty and potentially a great benefit in adding a prefilter $P$ subsequently to feedback design. On the other hand, the convexity of the design of $P$ can be exploited for an alternative “D-K” iteration scheme for the joint design of $F$ and $P$, as follows. The “D-K” iteration (see \cite{4}) alternates between an analysis stage that computes scales, and a controller synthesis stage which computes $P$ and $F$. In this work we show that for fixed $F$, the problem is jointly convex in the scales and the prefilter $P$. Thus an alternative iteration is possible, where $P$ is not “frozen” at the analysis stage.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{twoDegreeLayout.pdf}
\caption{Two-Degree-of-Freedom Controller}
\label{fig:twoDegreeLayout}
\end{figure}

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This extra freedom would be beneficial in obtaining a more efficient iteration, although it is difficult to make definitive statements about these non-convex problems.

The paper is organized as follows: the problem formulation is explained in Section 2, and background on robustness analysis methods is reviewed. Section 3 presents an infinite dimensional convex condition which applies to all cases, and explains how it may be used for design in the LTI case. In Section 4 the synthesis problem for LTV uncertainty $\mathcal{H}_\infty$ performance is solved exactly by an LMI condition. For the $\mathcal{H}_2$, LTV case, methods based on basis functions are discussed. Concluding remarks are given in Section 5.

# 2 Preliminaries

## 2.1 Notation
For state-space matrices $A$, $B$, $C$, $D$, the corresponding transfer function is denoted by

$$G(s) = C(s I - A)^{-1} B + D = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

For a partitioned operator $M \triangleq \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and two operators $P$ and $Q$ with compatible dimensions, the lower and upper linear fractional transformations (LFTs) are defined as

$$\mathcal{F}_l(M, Q) := M_{11} + M_{12} Q (I - M_{22} Q)^{-1} M_{21},$$

$$\mathcal{F}_u(M, P) := M_{22} + M_{21} P (I - M_{11} P)^{-1} M_{12}.$$  

## 2.2 Problem Formulation
Starting from the configuration of Figure 1, where $\Delta$ has the structure (1), we assume that a robustly stabilizing feedback $F$ has been chosen. This paper concerns the subsequent design of $P$ from the point of view of robust tracking. For this purpose we rewrite the problem in the standard form of Figure 2, in terms of a generalized plant

$$G(s) = \begin{bmatrix} A & B & E & M \\ C_1 & L_1 & H_1 & N_1 \\ C_2 & L_2 & H_2 & N_2 \\ 0 & 0 & I & 0 \end{bmatrix}.$$

![Figure 2: Standard form](image)

with $A \in \mathbb{R}^{n \times n}$ stable. Despite the aspect of Figure 2, there is no feedback: the prefilter $P$ sees the reference signal $y = r$, and produces the control input $u$. The performance output $z$ is the quantity we must keep small, and is different from $s$ in Figure 1; for example, $z$ may denote the tracking error with respect to a given desired response. The dimensions of the signals $p$, $r$, $u$, $q$, $z$ and $y$ are respectively $d_p$, $d_r$, $d_u$, $d_q$, $d_z$ and $d_y$.

We now specify the uncertainty $\Delta$: if $L_c(L_2)$ is the space of bounded, causal operators on $L_2$, then $\Delta$ will be restricted to the structured unit ball

$$B_{\Delta,LTV} = \{ \Delta \in L_c(L_2) : \| \Delta \| \leq 1, \Delta \text{ as in (1)} \}.$$  

The superindex $LTV$ denotes time-varying operators; $B_{\Delta,LTI}$ is the time-invariant subclass.

Since the feedback $F$ is stabilizing, it is sufficient to choose a stable $P$ to ensure overall stability. The transfer function between $(p, r)$ and $(q, z)$ is given by

$$T_F(s) := \mathcal{F}_l(G(s), P(s)).$$

Now given $\Delta$, the closed loop map from $r$ to $z$ is $\mathcal{F}_u(T_F, \Delta)$. For a prespecified $\gamma$, the system is said to have robust $\mathcal{H}_\infty$ performance level $\gamma$ if

$$\sup_{\Delta \in B_{\Delta,LTV}} \| \mathcal{F}_u(T_F, \Delta) \|_{\infty} < \gamma, \quad (3)$$

and robust $\mathcal{H}_2$ performance level $\gamma$ if

$$\sup_{\Delta \in B_{\Delta,LTV}} \| \mathcal{F}_u(T_F, \Delta) \|_2 < \gamma. \quad (4)$$

For LTI uncertainty, the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms in (3) and (4) are given their standard Hardy space interpretation. In the LTV case, the notation $\| \cdot \|_\infty$ is used for the $L_\infty$-induced norm; $\| \cdot \|_2$ is defined in terms of the response to deterministic white noise, as introduced in [13, 14].

The problem is to find $P$, if it exists, to satisfy the robust $\mathcal{H}_\infty$ ($\mathcal{H}_2$) performance requirements.

## 2.3 Analysis Conditions
Let us consider the hermitian scaling matrices

$$X = \text{diag}[X_1, \ldots, X_L, x_{L+1}^I, \ldots, x_{L+1}^F X_I], \quad (5)$$

which commute with the structure of $\Delta$. We denote by $X$ the set of positive definite matrices of this form. We state the following conditions:

**Condition 1** [$\mathcal{H}_\infty$]: There exists a scaling function $X(\omega)$ taking values in $X$ such that

$$T_F(j\omega)^* \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} T_F(j\omega) - \begin{bmatrix} X(\omega) & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0. \quad (6)$$
Condition 2 $[H_2]:$ There exists $X(\omega)$ taking values in $\mathcal{X}$, and $Y(\omega) = Y^*(\omega) \in C^d$, such that
\[
T_P(\omega)^* \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} T_P(\omega) - \begin{bmatrix} X(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} < 0, \quad (7)
\]
\[
\int_{-\infty}^{+\infty} \text{trace}(Y(\omega)) \frac{d\omega}{2\pi} < \gamma^2. \quad (8)
\]

The following properties are known [4, 11, 12, 13, 14, 19]: Conditions 1 and 2 are sufficient for robust $H_\infty$ ($H_2$) performance over $\mathcal{B}_{\Delta LTI}$. While they are in general conservative, it is known (see [14, 17]) that they become necessary and sufficient if LTI uncertainty is replaced by “arbitrarily slowly varying” uncertainty [17]. For this reason they provide an attractive method for robust performance evaluation for the LTI case. If $X(\omega)$ is constrained to be a constant $X \in \mathcal{X}$, then the conditions 1 and 2 are equivalent to the robust $H_\infty$ ($H_2$) performance over the class $\mathcal{B}_{\Delta LTV}$ [19, 11, 14].

3 Convex frequency domain conditions for prefilter synthesis

To obtain a convex characterization of the prefilter synthesis problem we first write an equivalent representation for the analysis conditions, obtained in [7].

It follows from (2) that
\[
T_P = \begin{bmatrix} T_0 & T_1 \\ T_0 + T_{12} P \end{bmatrix}, \quad (9)
\]
with $T_P$ partitioned in correspondence to the uncertainty and reference inputs. Let us define
\[
\Psi(\omega, X(\omega), Y(\omega), P(\omega)) :=
\begin{bmatrix}
T_0(j\omega)X(\omega)T_0(j\omega)^* - \begin{bmatrix} X(\omega) & 0 \\ 0 & 0 \end{bmatrix} T_{11}(j\omega) + T_{12}(j\omega) P(\omega) \\
T_{11}(j\omega) + T_{12}(j\omega) P(\omega) \end{bmatrix}
\]
\[
(10)
\]
Note that function $\Psi$ is affine in the unknowns $X(\omega)$, $Y(\omega)$, and $P(\omega)$. Next lemma follows by using (9) and the fact that the set $\mathcal{X}$ is closed under inversion [7].

Lemma 1 Condition (7) is equivalent to
\[
\Psi(\omega, X(\omega), Y(\omega), P(\omega)) < 0
\]
For the $H_\infty$ case, pick $Y(\omega) = \gamma^2 I$ and pose

Problem 1
\[
\min_{X(\omega), \gamma^2, P(\omega)} \gamma^2
\]
subject to $P(s)$ stable and
\[
\Psi(\omega, X(\omega), \gamma^2 I, P(\omega)) < 0 \quad \forall \omega. \quad (11)
\]

For the $H_2$ case, we have

Problem 2
\[
\min_{X(\omega), Y(\omega), P(\omega)} \int_{-\infty}^{+\infty} \text{trace}(Y(\omega)) \frac{d\omega}{2\pi} < \gamma^2
\]
subject to $P(s)$ stable and
\[
\Psi(\omega, X(\omega), Y(\omega), P(\omega)) < 0 \quad \forall \omega. \quad (12)
\]

These problems involve the minimization of a linear objective subject to convex, infinite dimensional constraints. The corresponding cost functions provide a bound for the square of the worst-case $H_\infty$ ($H_2$) norm in (3) (respectively (4)), from Lemma 1 and the conditions of Section 2.3. Also, the problems apply to both cases of constant and frequency dependent scalings $X(\omega)$ (LTV or LTI uncertainty).

If $X(\omega)$ in Problems 1 and 2 is allowed to vary arbitrarily in frequency, the conditions are inherently infinite dimensional. In practice one seeks a finite dimensional approximation. One way of approaching this is by frequency gridding; for example in the case of Problem 1, we would choose a set of frequencies $\omega_0, \ldots, \omega_N$ and write
\[
\Psi(\omega_i, X_i, \gamma^2 I, P_i) < 0 \quad i = 0, \ldots, N. \quad (13)
\]
This problem is decoupled across frequency except for the variable $\gamma^2$; an efficient approach to the computation would be to do a linear search in $\gamma$, and for each $\gamma$ test for the feasibility of the decoupled LMIs (13).

The $H_2$ problem is already decoupled: we can solve
\[
\min_{X_i, \gamma^2, P_i} \text{trace}(Y_i)
\]
subject to $X_i \in \mathcal{X}$ and
\[
\Psi(\omega_i, X_i, \gamma^2 I, P_i) < 0 \quad i = 0, \ldots, N,
\]
This means that in both cases a large number of frequency points can be handled with computation time growing linearly with $N$. The advantage of the frequency gridding approach is that insight can be gained into the ideal frequency response which is required of the prefilter. However, the grid offers only an approximation and no hard guarantees. In addition, another difficulty appears, since for given $P(\omega_i)$, we are faced with the interpolation problem of fitting a stable $P(s)$ to these points. In practice we could compute many points for $P(\omega_i)$ first, then construct a stable approximation of sufficient order. Alternatively, the stability constraint may be easily enforced by using basis functions, the alternative procedure to obtain a finite dimensional problem [2].
4 Finite dimensional LMI synthesis for the LTV uncertainty case

In this section we concentrate on the problem of prefilter design to satisfy Conditions 1 or 2 with constant scalings, which are necessary and sufficient for robust $\mathcal{H}_\infty$ ($\mathcal{H}_2$) performance under LTV uncertainty.

4.1 Exact solution for the $\mathcal{H}_\infty$ case

We will now show that the existence of dynamic prefilter that guarantee a given level $\gamma$ of robust $\mathcal{H}_\infty$ performance against LTV uncertainties is equivalent to a finite dimensional LMI in state space. For such purpose, let $\hat{X}_R$ denote a matrix whose columns constitute an orthonormal basis for the kernel of $[\begin{array}{cc} M' & N' \end{array}]$, and

$$\hat{L}_R \triangleq \left[ \begin{array}{cc} \hat{X}_R & 0 \\ 0 & I_{d_p+d_r} \end{array} \right].$$

**Theorem 2** There exists a dynamic prefilter $P(s)$ satisfying robust $\mathcal{H}_\infty$ performance level $\gamma$ under perturbations in $\Delta_{\text{LTV}}$ if and only if there exist symmetric, positive definite matrices $R \in \mathbb{R}^{n \times n}$, $Z \in \mathbb{R}^{n \times n}$, and $X \in \mathbb{R}^n$ satisfying

$$\hat{L}_R' \begin{bmatrix} AR + RA' & RC'Q^{-1} B X & E \\ CR & X' & 0 \\ X'B' & XL' & -X \\ E' & 0 & 0 \end{bmatrix} < 0,$$

$$R - Z \geq 0.$$  

**Proof**: Starting from Condition 1 for constant $X$, we replace $X$ by $X^{-1}$ for convenience, and introduce

$$Q := \begin{bmatrix} X^\frac{1}{2} & 0 \\ 0 & I \end{bmatrix}; \quad Q_\gamma := \begin{bmatrix} X^\frac{1}{2} & 0 \\ 0 & \gamma^{-1} I \end{bmatrix}.\tag{18}$$

Then (6) is equivalent to

$$\|Q^{-1} T_P Q_\gamma\|_\infty < 1.\tag{17}$$

Since $T_P = F_0(G, P)$, (17) can be rewritten as

$$\|F_0(G, Q, P)\|_\infty < 1,\tag{18}$$

where

$$G_Q := \begin{bmatrix} A & B E & M \\ Q^{-1} C & L H & Q_\gamma Q^{-1} N \\ 0 & 0 & Q_\gamma \end{bmatrix}.\tag{19}$$

Now (18) is a standard $\mathcal{H}_\infty$ synthesis problem. By applying to our case the LMI formulation of [5], condition (18) is feasible if and only if there exist matrices $R > 0$ and $S > 0$ such that

$$\hat{L}_R' \begin{bmatrix} AR + RA' & RC'Q^{-1} B X & E \\ CR & X' & 0 \\ X'B' & XL' & -X \\ E' & 0 & 0 \end{bmatrix} < 0,$$

$$R - Z \geq 0,$$

where

$$\hat{L}_R := \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{d_p+d_r} \end{bmatrix}; \quad \hat{L}_R' := \begin{bmatrix} I_n & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_{d_p+d_r} \end{bmatrix},$$

$$U = \begin{bmatrix} I_{d_p} \\ 0_{d_r \times d_p} \end{bmatrix}.$$
4.2 Basis function approach for the $\mathcal{H}_2$ case

Concerning the robust $\mathcal{H}_2$ performance in the LTV uncertainty case, it was found in [14, 15] that the $Y$ scaling can be eliminated in Condition 2, leading to an LMI condition. If the closed loop $T_p$ has realization

$$T_p = \begin{bmatrix} A & B_1 & B_2 \\ C & 0 & 0 \end{bmatrix},$$

then Condition 2 is equivalent to

**Condition 3** There exist $X \in X$, and hermitian matrices $\mathcal{P}_- > 0$, $\mathcal{P}_+$ and $Z$ such that

$$\begin{bmatrix} \mathcal{A}\mathcal{P}_- + \mathcal{P}_-\mathcal{A}^* + B_1XK_1 & \mathcal{P}_-C^* \\ C\mathcal{P}_- & -X_0I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \mathcal{A}\mathcal{P}_+ + \mathcal{P}_+\mathcal{A}^* + B_1XK_1 & \mathcal{P}_+C^* \\ C\mathcal{P}_+ & -X_0I \end{bmatrix} < 0,$$

$$\begin{bmatrix} Z & \mathcal{P}_+ & -\mathcal{P}_- \\ \mathcal{P}_+ & \mathcal{P}_+ & \mathcal{P}_- \end{bmatrix} > 0,$$

$$trace(Z) < \gamma^2.$$

It was observed in [7] that static prefilter synthesis reduces to an LMI. For the dynamic case, although there is some encouraging evidence we have not as yet obtained a complete solution. For this reason we will only discuss here the approximate method which results if we use a basis function for $P(s)$. A general representation for the span of certain stable basis functions is

$$P(s) = \begin{bmatrix} A & B_2 \\ C & D_p \end{bmatrix} \begin{bmatrix} B_p(\theta) \\ D_p(\theta) \end{bmatrix},$$

where $C_p$, $A_p$ are fixed and observable, $A_p$ stable, and $B_p$, $D_p$ depend linearly on a vector of parameters $\theta$. After some manipulation we obtain the following transfer function for the closed loop system:

$$T_p = \begin{bmatrix} A & MC_p & B & E + MD_p(\theta) \\ 0 & A_p & 0 & B_p(\theta) \\ C & 0 & 0 & 0 \end{bmatrix}.$$

Note that $A$, $C$ and $B_1$ are fixed in (26), while $B_2$ depends affinely on the parameters $\theta$. Substitution into Condition 3 gives an LMI in the unknowns $\mathcal{P}_-$, $\mathcal{P}_+$, $Z$, $\gamma^2$ and $\theta$, from which the problem can be solved.

5 Discussion and Conclusions

The results in this paper show that robust prefilter design for systems with structured uncertainty can be reduced to convex optimization problems. Infinite dimensional convex conditions apply to dynamic prefilter design under LTI uncertainty, and we have discussed methods to approximate them in finite dimensions. For LTV uncertainty, the problem is solved exactly via an LMI in the $\mathcal{H}_\infty$ performance case, and we have provided basis function methods for the $\mathcal{H}_2$ case.

References


