closed loop system -/-passive by solving the
is to say, one can construct a dynamical compensator that renders a
criterion of feedback systems, which is a sort of generalization of
the passivity (positivity) theorem and effective for robust stability
version of Theorem 18, a closed loop system with the desired amount
stabilizability result by output feedback. In the case of linear systems,
method of a y-passivity problem in nonlinear control systems, that
a transfer function. One can design, by making use of the linear
real systems can be a measure of maximum phase difference of
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In [4], the authors propose a static prefilter chosen as
In [2] the authors used H_\infty techniques for the feedback part design
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technique and a model following interconnection structure.

In [3] the concept of the one step H_\infty controller design was
introduced and applied. In this approach the two parts are jointly
determined which reduces the order of the resulting controller.

A good discussion about the characteristics of H_\infty, H_2, and
LQG/LTR two-degree-of-freedom controller design techniques is
encountered in [4]. In [4], the authors propose a static prefilter chosen
as the inverse of the steady-state closed-loop transfer function, provided
the plant is square. This design procedure assures decoupling and zero
error at steady state.

A different approach is used in [5] where the feedback controller is
obtained using eigenstructure assignment. In this case, a static prefilter
is designed from a least squares technique.

The requirements on P and F are different. While F is strongly
related to the robustness and disturbance attenuation properties, P is
related to the performance with respect to command inputs. From this
point of view it seems convenient to separately study the controller

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H_\infty and H_2 Design Techniques for a Class of Prefilters
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Abstract—This paper presents H_\infty and H_2 techniques for the design of
a certain class of prefilters. In order to avoid high order prefilters we
use a particular structure in which both prefilter and feedback controller
share the same dynamics. The suboptimal H_\infty design problem is shown
to be equivalent to an linear matrix inequality (LMI) constraint, and
the solution of the H_2 optimal design may be obtained by a simple formula.
The H_2 technique developed will be illustrated on a helicopter example
and compared with classical techniques.

I. INTRODUCTION

In this paper we consider the control system of Fig. 1 where w
denotes the disturbances, u the control action, s the performance
output, r the reference inputs, and v the measured outputs. The plant
G_p is controlled by a two-degree-of-freedom controller K which
is composed by the feedback part F and the prefilter P such that
u = Fv + Pr.

The feedback part F is classically designed from specifications
like stability, disturbance rejection, and robustness.

The prefilter P is introduced to improve the command response,
i.e., the transfer between r and s.

It was proven [1] that if the unique constraint on P and F is their
realizability, the stability and disturbances attenuation on the one hand
and command response on the other hand are independent require-
ments. When P and F are constrained in order that independence is
lost, a compromise will appear between the feedback controller and
the prefilter designs.

Several two-degree-of-freedom controllers synthesis methods have
been reported which have used different techniques such as H_2,
LQG/LTR, H_\infty, and eigenstructure assignment.

In [2] the authors used H_\infty techniques for the feedback part design
in a first step. Separately, the prefilter was designed using an H_\infty
technique and a model following interconnection structure.

In [3] the concept of the one step H_\infty controller design was
introduced and applied. In this approach the two parts are jointly
determined which reduces the order of the resulting controller.

A good discussion about the characteristics of H_\infty, H_2, and
LQG/LTR two-degree-of-freedom controller design techniques is en-
countered in [4]. In [4], the authors propose a static prefilter chosen
as the inverse of the steady-state closed-loop transfer function, provided
the plant is square. This design procedure assures decoupling and zero
error at steady state.

A different approach is used in [5] where the feedback controller is
obtained using eigenstructure assignment. In this case, a static prefilter
is designed from a least squares technique.

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related to the performance with respect to command inputs. From this
point of view it seems convenient to separately study the controller

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and prefilter design problems, even though a relative compromise will exist in general.

At this point it is important to notice the differences between \( H_\infty \) and \( H_2 \) design techniques. For systems with a relatively broad set of reference inputs, which may possess significant power within arbitrarily small bandwidths, \( H_\infty \) is clearly appropriate. When the references belongs to a small set of test inputs, like steps, pulses, or ramps, the \( H_2 \) norm is more appropriate as a performance index.

An important point in classical \( H_\infty \) and \( H_2 \) design techniques is that they lead to controllers with the same order of the generalized plant which includes, for a prefilter design, the open loop plant, the feedback controller, and auxiliary filters used for loop shaping purposes. In practical problems this may easily lead to a prefilter of unacceptably high order.

Despite many interesting results, the reduced-order \( H_\infty \) and \( H_2 \) synthesis problems are still difficult because they are, in general, nonconvex and computationally hard. Hence, the development of \( H_\infty \) and \( H_2 \) design techniques that are computationally attractive and lead to a low-order prefilter is an important issue.

Aiming to improve the response of the system in Fig. 1 to the reference input, we present in this paper \( H_\infty \) and \( H_2 \) techniques for the design of the prefilter \( P \). For such a purpose we assume that \( F \) was already determined and ensures the closed-loop stability and some additional performance requirements.

There are three main characteristics of the techniques presented in this paper. First, \( P \) and \( F \) are determined separately which allows us to consider a prefilter design technique independently of the technique chosen for the feedback controller design. Moreover, we will assume a particular structure for \( P \) such that the order of the overall system is the same with or without the prefilter. Finally, the design procedures are implemented with low computational cost. In fact, the suboptimal \( H_\infty \) design problem is shown to be equivalent to a linear matrix inequality (LMI) constraint on the prefilter free parameters. In addition, the solution of the optimal \( H_2 \) design is analytic and may be obtained by a simple formula. Alternatively, this problem may be solved as an LMI optimization problem.

The \( H_2 \) and \( H_\infty \) prefilter design problems and a motivation for the interconnection structure adopted are presented in the next section. Section III presents the main results of this work. These results are applied in Section IV to a helicopter model, and the Section V finishes the work with some concluding remarks.

**Notation:** The subspace spanned by the columns of the matrix \( A \) will be denoted as \( \text{Im}(A) \). \( T_{ab} \) denotes the transfer function from signal \( b \) to signal \( a \). The transfer function \( G(s) = C(sI-A)^{-1}B+D \) will be denoted as

\[
G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

II. PROBLEM STATEMENT

Let us suppose that the feedback controller \( F \) is determined by some convenient technique such that the closed-loop system has acceptable disturbance attenuation and robustness properties. The problem in this paper is to design the prefilter \( P \) in order to improve the performance of the system with respect to input reference signals.

Consider Fig. 2 where \( G_p \) denotes the plant model, \( K \) the controller, and \( M \) the target model which expresses the desired command response. The block \( W \) weights in the frequency domain, the error with respect to the target model. The prefilter \( F \) will be determined in order to minimize that error in the \( H_\infty \) or \( H_2 \) sense.

This model-following structure admits a standard representation in the form of Fig. 3, where the generalized plant \( G \) has a particular structure

\[
\begin{aligned}
\dot{z} &= A z + B_1 r + B_2 u \\
z &= C_1 z + D_{11} r + D_{12} u \\
y &= \begin{bmatrix} r \\ v \end{bmatrix} = \begin{bmatrix} C_{20} z + D_{22} u \\ D_{21} u \end{bmatrix}
\end{aligned}
\]

where \( z(t) \in \mathbb{R}^n, r(t) \in \mathbb{R}^m, u(t) \in \mathbb{R}^m, z(t) \in \mathbb{R}^m, v \in \mathbb{R}^m \).

In the standard notation

\[
G(s) = \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ D_{11} & D_{12} \\ 0 & 0 \end{bmatrix}
\]

with

\[
C_2 = \begin{bmatrix} 0 \\ C_{20} \\ 0 \end{bmatrix}, \quad D_{21} = I, \quad D_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

For simplicity let us assume that \( D_{220} = 0 \).

The controller structure is the following:

\[
K(s) = \begin{bmatrix} A_F \\ C_F \end{bmatrix} \begin{bmatrix} B_r & B_o \\ D_r & D_o \end{bmatrix}
\]

where \( A_F \in \mathbb{R}^{n_x \times n_x} \). The \( P \) and \( F \) parts are included in the \( K \) structure in the following way:

\[
u(s) = F(s)v(s) + P(s)r(s)
\]

\[
F(s) = \begin{bmatrix} A_F & B_r \\ C_F & D_r \end{bmatrix}, \quad P(s) = \begin{bmatrix} A_F & B_o \\ C_F & D_o \end{bmatrix}
\]

Note that \( F(s) \) is assumed to be given, and \( P(s) \) has a particular structure since \( P \) and \( F \) have the same dynamic matrix \( A_F \) and output matrix \( C_F \). The parameters to be determined are \( B_r \) and \( D_r \). By tuning these parameters we are modifying the zero location and static response of the prefilter; in contrast, the poles are invariant with
III. MAIN RESULTS

Consider the controller $K(s)$ given by (1). The closed-loop system of Figs. 2 and 3 is

$$T_{cr} = \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$$

with

$$A_{cl} = \begin{bmatrix} A + B_{2} D_{12} C_{2o} & B_{2} C_{F} \\ B_{o} C_{2o} & A_{F} \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} B_{1} + B_{2} D_{12} \\ B_{r} \end{bmatrix} = E + M R,$$

$$C_{cl} = \begin{bmatrix} C_{1} + D_{12} D_{12} C_{2o} \\ D_{12} C_{F} \end{bmatrix}, \quad D_{cl} = D_{11} + D_{12} D_{r} = H + N R.$$  

For convenience we introduced in the above equations the following constant matrices:

$$E \triangleq \begin{bmatrix} B_{1} \\ 0 \end{bmatrix}, \quad M \triangleq \begin{bmatrix} 0 & B_{2} \\ I & 0 \end{bmatrix},$$

$$H \triangleq D_{11}, \quad N \triangleq \begin{bmatrix} 0 & D_{12} \end{bmatrix}.$$  

and the following variable matrix that contains the free prefilter parameters:

$$R \triangleq \begin{bmatrix} B_{r} \\ D_{r} \end{bmatrix}.$$  

It is worth noting that $A_{cl}$ and $C_{cl}$ are constant, i.e., they do not depend on the prefilter free parameters.

Let us consider the singular value decomposition of $D_{12} \in R^{d_{2} \times d_{u}}$

$$D_{12} = U \Sigma V' = [U_{1} \quad U_{2}] \begin{bmatrix} \Sigma_{p} & 0 \\ 0 & 0 \end{bmatrix} [V_{1}' \quad V_{2}']$$

$$= U_{1} \Sigma_{p} V_{1}'$$  

where $U \in R^{d_{2} \times d_{1}}$ and $V \in R^{d_{0} \times d_{u}}$ are unitary matrices, and $\Sigma_{p} \succ 0$. By denoting $p = \text{rank}(D_{12}), U_{1} \in R^{d_{2} \times p}$ and $V_{1} \in R^{d_{u} \times p}$.

A. $H_{\infty}$ Case

1) Assumptions:

$\bullet$ $A_{cl}$ is asymptotically stable.

Proposition 1: The $H_{\infty}$ suboptimal prefilter synthesis problem has a solution if and only if there exists a symmetric matrix $Y \succ 0$ with dimensions $(n+n_{k})(n+n_{k})$, and $R \in R^{d_{k} \times d_{r}} \times d_{r}$ such that

$$\begin{bmatrix} A_{cl}' Y + A_{cl} Y & E + M R & Y C_{cl}' \\ E' + R' M' & -\gamma I & H' + R' N' \\ C_{cl} Y & H + N R & -\gamma I \end{bmatrix} < 0.$$  

(B) By the Bounded Real Lemma [9] the stated problem is equivalent to the existence of a symmetric matrix $X_{cl} > 0$ such that

$$\begin{bmatrix} A_{cl}' X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} \\ B_{cl}' X_{cl} & -\gamma I & C_{cl}' \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0.$$  

Denote $Y = X_{cl}^{-1}$. Multiplying the left and right sides by the positive definite matrix

$$\begin{bmatrix} Y & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and considering the definitions (3) and (4), after some manipulation we get (7). The prefilter parameters are obtained from $R$ by (5).

Expression (7) involves an LMI feasibility problem in the $\gamma, Y$ and $R$ parameters, and is numerically solvable in a very efficient manner. The $H_{\infty}$ optimal prefilter synthesis may be formulated, then, as a standard problem of linear objective minimization subject to an LMI constraint.

B. $H_{2}$ Case

1) Assumptions:

$\bullet$ $A_{cl}$ is asymptotically stable.

$\bullet$ $(C_{cl}, A_{cl})$ is completely observable.

$\bullet$ $\text{Im}(D_{12}) \subseteq \text{Im}(D_{11})$.

The last assumption is necessary and sufficient to ensure that $R$ exists such that the closed-loop system is strictly proper.

For the closed-loop system (2), if $D_{cl} = 0$, we have

$$\|T_{cr}\|_{2}^{2} = \text{Tr}(B_{cl}' L_{o} B_{cl}) = \text{Tr}(E + M R)' L_{o}(E + M R)$$  

where $L_{o}$ is the observability grammian. It is well known that $L_{o} > 0$ satisfies the Lyapunov equation

$$A_{cl}' L_{o} + L_{o} A_{cl} + C_{cl}' C_{cl} = 0.$$  

It is worth noting that $A_{cl}$ and $C_{cl}$ are constant, i.e., they do not depend on the prefilter free parameters.

Let us consider the singular value decomposition of $D_{12} \in R^{d_{2} \times d_{u}}$

$$D_{12} = U \Sigma V' = [U_{1} \quad U_{2}] \begin{bmatrix} \Sigma_{p} & 0 \\ 0 & 0 \end{bmatrix} [V_{1}' \quad V_{2}']$$

$$= U_{1} \Sigma_{p} V_{1}'$$  

where $U \in R^{d_{2} \times d_{1}}$ and $V \in R^{d_{0} \times d_{u}}$ are unitary matrices, and $\Sigma_{p} \succ 0$. By denoting $p = \text{rank}(D_{12}), U_{1} \in R^{d_{2} \times p}$ and $V_{1} \in R^{d_{u} \times p}$.

A. $H_{\infty}$ Case

1) Assumptions:

$\bullet$ $A_{cl}$ is asymptotically stable.

Proposition 1: The $H_{\infty}$ suboptimal prefilter synthesis problem has a solution if and only if there exists a symmetric matrix $Y \succ 0$ with dimensions $(n+n_{k})(n+n_{k})$, and $R \in R^{d_{k} \times d_{r}} \times d_{r}$ such that

$$\begin{bmatrix} A_{cl}' Y + A_{cl} Y & E + M R & Y C_{cl}' \\ E' + R' M' & -\gamma I & H' + R' N' \\ C_{cl} Y & H + N R & -\gamma I \end{bmatrix} < 0.$$  

(B) By the Bounded Real Lemma [9] the stated problem is equivalent to the existence of a symmetric matrix $X_{cl} > 0$ such that

$$\begin{bmatrix} A_{cl}' X_{cl} + X_{cl} A_{cl} & X_{cl} B_{cl} \\ B_{cl}' X_{cl} & -\gamma I & C_{cl}' \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0.$$  

Denote $Y = X_{cl}^{-1}$. Multiplying the left and right sides by the positive definite matrix

$$\begin{bmatrix} Y & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and considering the definitions (3) and (4), after some manipulation we get (7). The prefilter parameters are obtained from $R$ by (5).

Expression (7) involves an LMI feasibility problem in the $\gamma, Y$ and $R$ parameters, and is numerically solvable in a very efficient manner. The $H_{\infty}$ optimal prefilter synthesis may be formulated, then, as a standard problem of linear objective minimization subject to an LMI constraint.

B. $H_{2}$ Case

1) Assumptions:

$\bullet$ $A_{cl}$ is asymptotically stable.

$\bullet$ $(C_{cl}, A_{cl})$ is completely observable.

$\bullet$ $\text{Im}(D_{12}) \subseteq \text{Im}(D_{11})$.

The last assumption is necessary and sufficient to ensure that $R$ exists such that the closed-loop system is strictly proper.

For the closed-loop system (2), if $D_{cl} = 0$, we have

$$\|T_{cr}\|_{2}^{2} = \text{Tr}(B_{cl}' L_{o} B_{cl}) = \text{Tr}(E + M R)' L_{o}(E + M R)$$  

where $L_{o}$ is the observability grammian. It is well known that $L_{o} > 0$ satisfies the Lyapunov equation

$$A_{cl}' L_{o} + L_{o} A_{cl} + C_{cl}' C_{cl} = 0.$$  

Note that $L_{o}$ does not depend on the prefilter free parameters.

Since (8) is a convex function on the parameter $R$, it is possible to describe the set of prefilters that assure $\|T_{cr}\|_{2} < \beta$ as a convex set. The next proposition gives an LMI formulation for this set, and follows from (8) and a well known algebraic manipulation [7].

Proposition 2: There exists a controller $K$ compatible with (1) such that $\|T_{cr}\|_{2} < \beta$ if and only if there exists a symmetric matrix $X \in R^{d_{r} \times d_{r}}$ and $R \in R^{d_{k} \times d_{r}} \times d_{r}$ such that

$$\begin{bmatrix} \text{Tr}(X) < \beta^{2} \\ (E + M R)' L_{o}^{-1} \\ H + N R = 0 \end{bmatrix} > 0.$$  

Equation (10) involves an LMI feasibility problem in $\beta^{2}, X$, and $R$. The equality constraint in (10) reduces the order of the LMI since it allows us eliminate several free elements of $R$.

Propositions 1 and 2 allow us to design the prefilter with considerable freedom. It is possible to solve any one of the following optimal synthesis problems:

- $H_{\infty}$ problem: $\min_{R} \|T_{cr}\|_{\infty}$.
- $H_{2}$ problem: $\min_{R} \|T_{cr}\|_{2}$.
- $H_{\infty}/H_{2}$ mixed problem: $\min_{R} \|T_{cr}\|_{2}$ subject to $\|T_{cr}\|_{\infty} < \gamma$.
- $H_{\infty}/H_{2}$ mixed problem: $\min_{R} \|T_{cr}\|_{\infty}$ subject to $\|T_{cr}\|_{2} < \beta$.  

The last two problems involve minimizing \( \beta(\gamma) \) simultaneously constrained by (7) and (10). The stated \( \mathcal{H}_2/\mathcal{H}_\infty \) mixed problem allows us to design optimum prefilters with respect to the \( \text{IFI}/\text{HR} \) sense.

The next proposition shows that it is possible to compute analytically the optimum prefilter in the \( \text{IFI}/\text{HR} \) sense.

**Proposition 3:** Assume \( B_2 \) has full column rank. Then, the design parameter \( R \) that minimizes \( \text{IFI}_Z \) is unique and is given by

\[
R = - \begin{bmatrix} 0 & V_1' \\ V_2' & M'L_o M \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_p^{-1} U_1' H \\ \Sigma_p^{-1} U_2' \end{bmatrix},
\]

where the grammian \( L_o \) is the unique solution of the Lyapunov equation (9).

**Proof:** The minimum of the quadratic form \( \text{Tr}(B_2^r L_o B_2^c) \) subject to \( H + NR = 0 \) is achieved for \( R \) and \( \Lambda \in \mathbb{R}^{p \times d} \) such that

\[
\frac{d}{dt} \text{Tr}[(E + M\Lambda)' L_o (E + M\Lambda) + \Lambda(H + NR)] = 0
\]

which yields

\[
2M'L_o(E + M\Lambda) + N'\Lambda' = 0 \quad (H + NR = 0).
\]

\( M \) has full column rank because it was assumed that \( B_2 \) has full column rank [see (3)]. Moreover, the grammian \( L_o \) is positive definite since the system is observable. Then, the matrix \( M'L_o M \) is positive definite too, and its inverse does exist.

By using (6) in the first equation in (12)

\[
2M'L_o(E + M\Lambda) + \begin{bmatrix} 0 & V_1' \\ V_2' & M'L_o M \end{bmatrix} \Lambda' = 0.
\]

By premultiplying by the nonsingular matrix \( T_1 \in \mathbb{R}^{(n_k + d_u) \times (n_k + d_u)} \) defined as

\[
T_1 = \begin{bmatrix} 0 & V_1' \\ 0 & V_2' \\ I & 0 \end{bmatrix},
\]

we have

\[
2T_1 M'L_o [M\Lambda + E] + \begin{bmatrix} \Sigma_p U_1' \Lambda' \\ 0 \\ 0 \end{bmatrix} = 0. \quad (13)
\]

The top block of (13) does not represent any constraint on \( R \) since the term \( \Sigma_p U_1' \Lambda' \) may assume any value in \( \mathbb{R}^{p \times d_f} \). Hence, we do not lose any solution if we get only the two lower blocks. Then, let us premultiply (13) by the matrix \( T_2 \in \mathbb{R}^{(n_k + d_u - p) \times (n_k + d_u)} \)

\[
T_2 = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \end{bmatrix}.
\]

Thus

\[
\begin{bmatrix} 0 & V_1' \\ I & 0 \end{bmatrix} M'L_o [M\Lambda + E] = 0. \quad (14)
\]

The second equation in (12) represents the condition that causes the system to be strictly proper. By using (4) and (6) we get

\[
H + [0 \quad U_1 \Sigma_p V_1'] R = 0.
\]

By premultiplying by \( U' \)

\[
\begin{bmatrix} U_1' \\ U_2' \end{bmatrix} [H + [0 \quad U_1 \Sigma_p V_1'] R] = 0
\]

we get

\[
U_1' H + [0 \quad \Sigma_p V_1'] R = 0 \quad (15)
\]

\[
U_2' H = 0.
\]

Equation (15) is equivalent to the assumption \( \text{Im}(D_{11}) \subset \text{Im}(D_{12}) \).

From (15) and (14) we get

\[
\begin{bmatrix} 0 & V_1' \\ I & 0 \end{bmatrix} M'L_o M \quad R = - \begin{bmatrix} 0 & V_2' \\ I & 0 \end{bmatrix} M'L_o E. \quad (16)
\]

Let us show that the matrix

\[
T = \begin{bmatrix} 0 & V_1' \\ V_2' & 0 \end{bmatrix} M'L_o M
\]

is nonsingular. For all \( w \in \text{Ker}(T) \) we have

\[
[0 \quad V_1'] w = 0.
\]

Thus exists \( y \in \mathbb{R}^{n_k + d_w} \) such that

\[
y = \begin{bmatrix} 0 \\ V_2' \end{bmatrix} y.
\]

Hence, \( w \in \text{Ker}(T) \) if and only if there exists \( y \) such that

\[
\begin{bmatrix} 0 & V_2' \\ I & 0 \end{bmatrix} M'L_o M \begin{bmatrix} 0 \\ V_2' \end{bmatrix} y = 0. \quad (18)
\]

Since \( V_2 \) has full column rank and \( M'L_o M \) is nonsingular, the matrix in (18) is nonsingular, too. Hence, \( \text{Ker}(T) = 0 \), \( T \) is invertible and (11) follows from (16).

**Corollary 1:** If \( D_{11} = 0 \) and \( D_{12} = 0 \) the optimal solution is given by

\[
R = -(M'L_o M)^{-1}M'L_o E. \quad (19)
\]

**IV. NUMERICAL EXAMPLE**

Now we will apply the previous results to a realistic example. Let us consider the eighth-order model presented in [10]. It is a rigid body model of a helicopter operating near hover and includes, in a residualized form, the main rotor’s dynamics.

The open-loop plant is unstable.

The model is adimensional, i.e., all the states and inputs are scaled conveniently for comparison purposes. For the state-space representation and a full description, see [10].

The input has four components: main rotor collective pitch, lateral cyclic pitch, longitudinal cyclic pitch, and tail rotor collective pitch. The measured outputs are heave rate \( w \), roll angle \( \phi \), pitch angle \( \theta \), and yaw rate \( \tau \).

The control objectives are the following:

- Second-order response in pitch and roll angles with damping factor greater than 0.7.
- First-order response in yaw and heave velocities.
- Bandwidths greater than 3.5 rad/s.
- Cross couplings in the step responses less than 10% in the first four seconds.

**A. \( \mathcal{H}_\infty \) Feedback Controller Design**

The interconnection structure adopted is shown in Fig. 4. \( W_1 \) weights the goodness of the tracking characteristics at low frequencies
and the disturbance attenuation at the output of the plant. $W_2$ weights the control action, and $W_2$ is used to limit the bandwidth for robustness reasons. The selected weights are

$$
W_1 = 15 \text{diag}(w_{1u}, w_{1\phi}, w_{1\theta}, w_{1\tau})
$$

$$
w_{1u} = \frac{1}{s + 0.1}
$$

$$
w_{1\phi} = w_{1\theta} = \frac{0.04}{s^2 + 2.0828s + 0.025}
$$

$$
w_{2u} = \text{diag}(w_{2u}, w_{2\phi}, w_{2\theta}, w_{2\tau})
$$

$$
w_{2u} = \frac{1}{s^2 + 424s + 56250}
$$

$$
w_{2\phi} = \frac{1}{s^2 + 424s + 56250}
$$

$$
W_3 = 0.01I_4
$$

The closed-loop frequency responses for yaw rate and roll angle are plotted in Figs. 5 and 6. The bandwidths are adequate, but there is a considerable resonance in both channels. This resonance is a consequence of the unstable open loop poles, and no feedback controller can suppress it completely. In addition, the responses decay 60 dB/dec.

The temporal response to a unitary step in roll command is plotted in Fig. 7. All the states are shown simultaneously. The response exhibits an important overshoot, about 60%, and cross couplings of 10%. The response characteristics for the other input channels are similar. 

### B. Prefilter Design

Since the specifications are in terms of the step response, an $\mathcal{H}_2$ technique seems to be appropriate to the prefilter design. Note that the controller $F$ is determined and coincides with $-C$. $C$ is the series controller of Fig. 4. The target model $M$ of Fig. 2 was determined from the specifications in a direct manner. A similar target model is used in [5].

$$
M(s) = \text{diag}(m_w, m_\phi, m_\theta, m_r)
$$

with

$$
m_w = m_\tau = \frac{4}{s + 4}
$$

$$
m_\phi = m_\theta = \frac{8}{s^2 + 4s + 4s + 8}.
$$

The weight $W$ was chosen as the identity matrix.

The prefilter parameters were obtained from (19). The resulting frequency responses are plotted in Figs. 5 and 6. Note that the
resonance was totally suppressed and the 20 dB/dec and 40 dB/dec rolloff for yaw rate and roll angle were achieved. There is a small prescaling. The overshoot in the step response was eliminated, as with a stronger low-frequency weighting in order to achieve using different techniques for the prefilter design. As was stated previously, the cross couplings were drastically reduced. The resonance in the frequency response was suppressed which is not possible to achieve with a feedback controller when the plant has unstable open loop poles. The performance levels achievable with our design techniques were compared with classical designs, and with reduced-order controllers.

V. CONCLUSION

$\mathcal{H}_\infty$ and $\mathcal{H}_2$ techniques for prefilters design were presented, and the usefulness of the proposed techniques was illustrated with a helicopter control problem. The design techniques are suited to improve the performance of the system with respect to the reference inputs. The prefilter structure is more powerful than those presented in [4] and [5] because the prefilter itself has two degrees of freedom, $B_r$ and $D_r$. An interesting point is that the order of the two-degree-of-freedom controller is not increased by the prefilter. Moreover, complete freedom exists to choose the target model $\tilde{M}$ and weighting filter $\tilde{W}$ of Fig. 2 since their order does not add to the controller. In fact, the order of the overall system is the same with or without the prefilter. The proposed design procedure applied to a helicopter control problem leads to a significant improvement of the tracking behavior. The overshoots in the step responses were eliminated and the cross couplings were reduced. The resonance in the frequency response was suppressed which is not possible to achieve with a feedback controller when the plant has unstable open loop poles. The performance levels achievable with our design techniques were compared with classical designs, and with reduced-order controllers.

REFERENCES